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Invariant Imbedding and Scattering Processes

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1. INTRODUCTION

A physical process is called *scattering* when a disturbance impinging on an obstacle is reflected, transmitted, or absorbed. The disturbance, for example, may be waves or particles which propagate through the obstacle. Typical processes are deep-rooted in many physical problems, such as transmission line problems, the diffusion of light by the atmosphere, radiative transfer, neutron diffusion, and certain problems of probability. From the point of view of reflection and transmission operators, the above physical systems are all governed by the same basic mathematical model. However, one often fails to recognize this unity because the different problems are seldom considered together.

There is a long history of work done in this area, by using the principle of Invariant Imbedding, for particular physical systems which is associated with the names of Ambarzumian, Chandrasekhar, Bellman, Kalaba, Redheffer, Preisendorfer, Ueno, Wing, and others (see References). A historical summary can be found in [1] [2] and is not repeated here. The treatment having closest affinity with ours is that of Redheffer [3] [1], of which this is a continuation.

In the present work Redheffer's result is extended to the time-dependent case. Generalized equations are presented in *local* and *state* forms. The local form describes the local behavior of internal intensities in the scattering process, while the state form pertains to reflection and transmission operators that involve the state of the obstacle as a whole. The generalized equations are applied to physical problems of great diversity and practical interest. These include Wing's equation for particles moving in a rod, a generalization of Chandrasekhar's equation for radiative transfer from the stationary to the time-dependent case, Boltzmann's transport equation, Bellman's equation for photon-diffusion and a time-dependent version of homogeneous diffusion equations.

2. Time-Dependent Scattering Processes

The abstract space we shall consider is the same as the one discussed in the stationary case (see [1] [2]) with the exception that elements of Hilbert space now are time-dependent, i.e., intensities are functions of time, T. To show this dependence explicitly we write $I_+(x;\, T)$, and $I_-(x;\, T)$, corresponding to the notation $I_+(x)$, and $I_-(x)$, in the stationary case. While our operators in the stationary case transformed $I_+(x)$ into $I_+(y)$, for example, they now transform $I_+(x;\, T_1)$ into $I_+(y;\, T)$.

For a given obstacle (x, y) the transmission operators t, τ, and reflection operators ρ, r, involve four parameters $(x, y;\, T,\, T_1)$. The action of these operators is described by the rule

$$I_+(y;\, T) = \int_{-\infty}^{\infty} t(x, y;\, T,\, T_1)\, I_+(x;\, T_1)\, dT_1 \tag{2.1}$$

and similarly in other cases. Roughly speaking, $t(x, y;\, T,\, T_1)$ describes the contribution at y, at time T, due to a unit incident intensity at x and time T_1. The above integral formula is the principle of superposition for linear processes. As Preisendorfer [4], we shall use a summation convention, namely, we agree that a repeated time variable is always to be integrated from $-\infty$ to ∞. The formula (2.1) is thus written

$$I_+(y;\, T) = t(x, y;\, T,\, T_1) I_+(x;\, T_1).$$

Naturally, similar notations apply to r, ρ, and t. Thus, we have

$$\begin{pmatrix} I_+(y;\, T) \\ I_-(x;\, T) \end{pmatrix} = \begin{pmatrix} t(x, y;\, T,\, T_1) & \rho(x, y;\, T,\, T_1) \\ r(x, y;\, T,\, T_1) & \tau(x, y;\, T,\, T_1) \end{pmatrix} \begin{pmatrix} I_+(x;\, T_1) \\ I_-(y;\, T_1) \end{pmatrix}. \tag{2.2}$$

The 2×2 matrix in (2.2) is the time-dependent scattering matrix and is written as $S(x, y;\, T,\, T_1)$. On physical grounds we assume, for $T \geqslant T_1$,

$$S(x, x;\, T,\, T_1) = \begin{pmatrix} \delta(T - T_1) & 0 \\ 0 & \delta(T - T_1) \end{pmatrix}. \tag{2.3}$$

But a stronger condition is needed, namely, we assume existence of the limits,

$$\begin{pmatrix} b_1(x;\, T,\, T_1) & a(x;\, T,\, T_1) \\ c(x;\, T,\, T_1) & b_2(x;\, T,\, T_1) \end{pmatrix}$$
$$= \lim_{\Delta \to 0+} \frac{1}{\Delta} \left[S\left(x, x + \Delta;\, T + \frac{\Delta}{\lambda},\, T_1\right) - S(x, x;\, T,\, T_1) \right], \tag{2.4}$$

where Δ/λ is the time delay required for the intensity to travel through a thin obstacle $(x, x + \Delta)$ and $\lambda = \lambda(x) > 0$ is the propagation speed at x.

For the stationary case, we have $\lambda = \infty$. The precise sense in which the limit is understood is that

$$\lim_{\Delta \to 0+} \frac{1}{\Delta} \rho \left(x, x + \Delta; T + \frac{\Delta}{\lambda}, T_1\right) I_-(x; T_1) = a(x; T, T_1) I_-(x; T_1)$$

for all intensities $I_-(x; T_1)$ in our Hilbert space, and similarly in other cases. We also assume that a, b_1, b_2, and c depend continuously on their arguments.

As shown in [2], the stationary problem is a special case of our time-dependent problem, with operators of the special forms

$$S(x, y; T, T_1) = S(x, y) \cdot \delta(T - T_1).$$

With the above definition of time-dependent operators and coefficients, we are ready to derive the local and state forms.

Let us begin our analysis by thinking of a thin obstacle (x, y) where $y = x + \Delta$. By Eq. (2.2) we write

$$\begin{aligned} I_+ \left(y; T + \frac{\Delta}{\lambda}\right) &= t \left(x, y; T + \frac{\Delta}{\lambda}, T_1\right) I_+(x; T_1) \\ &\quad + \rho \left(x, y; T + \frac{\Delta}{\lambda}, T_1\right) I_-(y; T_1). \end{aligned}$$

Subtracting $I_+(x; T)$ from both sides, we have

$$\begin{aligned} I_+ \left(y; T + \frac{\Delta}{\lambda}\right) - I_+(x; T) &= \left[t \left(x, y; T + \frac{\Delta}{\lambda}, T_1\right) - \delta(T - T_1)\right] I_+(x; T_1) \\ &\quad + \rho \left(x, y; T + \frac{\Delta}{\lambda}, T_1\right) I_-(y; T_1). \end{aligned}$$

Dividing by Δ and taking the limit, we obtain

$$\left(\frac{\partial}{\partial x} + \frac{1}{\lambda} \frac{\partial}{\partial T}\right) I_+(x; T) = b_1(x; T, T_1) I_+(x; T_1) + a(x; T, T_1) I_-(x; T_1).$$

In a similar manner, we obtain

$$\left(-\frac{\partial}{\partial x} + \frac{1}{\lambda} \frac{\partial}{\partial T}\right) I(x; T) = c(x; T, T_1) I_+(x; T_1) + b_2(x; T, T_1) I_-(x; T_1).$$

By combining the above two equations, we obtain the *generalized local form*,

$$\begin{pmatrix} \left(\frac{\partial}{\partial x} + \frac{1}{\lambda} \frac{\partial}{\partial T}\right) I_+(x; T) \\ \left(\frac{\partial}{\partial x} - \frac{1}{\lambda} \frac{\partial}{\partial T}\right) I_-(x; T) \end{pmatrix} = \begin{pmatrix} b_1(x; T, T_1) & a(z; T, T_1) \\ -c(x; T, T_1) & -b_2(x; T, T_1) \end{pmatrix} \begin{pmatrix} I_+(x; T_1) \\ I_-(x; T_1) \end{pmatrix}. \tag{2.5}$$

We shall continue our analysis and obtain our ***time-dependent state form*** from the result of the previous section; that is, from the time-dependent local form. To this end let us consider an obstacle $(x, y + \Delta)$ with an input $I_-(y + \Delta; T_1)$ at $y + \Delta$ only. Then by our definition of ρ we have

$$\rho\left(x, y + \Delta; T + \frac{\Delta}{\lambda}, T_1 - \frac{\Delta}{\lambda}\right) I_-\left(y + \Delta; T_1 - \frac{\Delta}{\lambda}\right)$$
$$= I_+\left(y + \Delta; T + \frac{\Delta}{\lambda}\right)$$

and

$$\rho(x, y; T, T_1) I_-(y; T_1) = I_+(y; T).$$

Those two equations, combined with the equations of local form, become

$$I_+\left(y + \Delta; T + \frac{\Delta}{\lambda}\right)$$
$$= I_+(y; T) + [b_1(y; T, T_1) I_+(y; T_1) + a(y; T, T_1) I_-(y; T_1)]\, \Delta + 0(\Delta)$$

and

$$I_-(y; T_1) = I_-\left(y + \Delta; T_1 - \frac{\Delta}{\lambda}\right)$$
$$+ [c(y; T_1, T) I_+(y; T) + b_2(y; T_1, T) I_-(y; T)]\, \Delta + 0(\Delta),$$

to give, within $0(\Delta)$,

$$\rho\left(x, y + \Delta; T + \frac{\Delta}{\lambda}, T_1 - \frac{\Delta}{\lambda}\right) I_-\left(y + \Delta; T_1 - \frac{\Delta}{\lambda}\right)$$
$$= I_+(y; T) + [b_1(y; T, T_2) \rho(x, y; T_2, T_1) I_-(y; T_1) + a(y; T, T_1) I_-(y; T_1)]\Delta \tag{2.6}$$

and

$$I_+(y; T) = \rho(x, y; T, T_1) I_-\left(y + \Delta; T_1 - \frac{\Delta}{\lambda}\right)$$
$$+ [\rho(x, y; T, T_3) c(y; T_3, T_2) \rho(x, y; T_2, T_1) I_-(y; T_1)$$
$$+ \rho(x, y; T, T_2) b_2(y; T_2, T_1) I_-(y; T_1)]\, \Delta. \tag{2.7}$$

By substituting (2.7) into (2.6) and subtracting

$$\rho(x, y; T, T_1)\, I_-\left(y + \Delta; T_1 - \frac{\Delta}{\lambda}\right)$$

from both sides of the equation, we have

$$\left[\rho\left(x, y+\Delta; T+\frac{\Delta}{\lambda}, T_1-\frac{\Delta}{\lambda}\right)-\rho(x, y; T, T_1)\right] I_-\left(y+\Delta; T_1-\frac{\Delta}{\lambda}\right)$$
$$=[a(y; T, T_1)+b_1(y; T, T_2)\rho(x, y; T_2, T_1)+\rho(x, y; T, T_2)b_2(y; T_2, T\ )$$
$$+\rho(x, y; T, T_3)c(y; T_3, T_2)\rho(x, y; T_2, T_1)]\, I_-(y; T_1)\,\Delta.$$

Then, upon taking the limit of the above equation, we obtain

$$\left(\frac{\partial}{\partial y}+\frac{1}{\lambda}\frac{1}{\partial T}-\frac{1}{\lambda}\frac{\partial}{\partial T_1}\right)\rho(x, y; T, T_1)$$
$$=a(y; T, T_1)+b_1(y; T, T_2)\rho(x, y; T_2, T_1)+\rho(x, y; T, T_2)b_2(y; T_2, T_1)$$
$$+\rho(x, y; T, T_3)c(y; T_3, T_2)\rho(x, y; T_2, T_1).$$

The derivation of the differential equations for operators t, τ and r is not presented here. But we merely state our results. The ***time-dependent state form*** (right) can be presented in the following brief form:

$$\frac{\partial}{\partial y}\begin{pmatrix} t & \rho \\ r & \tau \end{pmatrix}+\frac{1}{\lambda}\frac{\partial}{\partial T}\begin{pmatrix} t & \rho \\ 0 & 0 \end{pmatrix}-\frac{1}{\lambda}\frac{\partial}{\partial T_1}\begin{pmatrix} 0 & \rho \\ 0 & \tau \end{pmatrix}$$
$$=\begin{pmatrix} (b_1+\rho c)t & a+b_1\rho+\rho b_2+\rho c\rho \\ \tau c t & \tau(b_2+c\rho) \end{pmatrix}. \tag{2.8}$$

On the other hand, we can consider an obstacle $(x-\Delta, y)$ and by processes similar to the above, we obtain the *time-dependent state form* (left), namely,

$$-\frac{\partial}{\partial x}\begin{pmatrix} t & \rho \\ r & \tau \end{pmatrix}+\frac{1}{\lambda}\frac{\partial}{\partial T}\begin{pmatrix} 0 & 0 \\ r & \tau \end{pmatrix}-\frac{1}{\lambda}\frac{\partial}{\partial T_1}\begin{pmatrix} t & 0 \\ r & 0 \end{pmatrix}$$
$$=\begin{pmatrix} t(b_1+ar) & ta\tau \\ c+b_2r+rb_1+rar & (b_2+ra)\tau \end{pmatrix}. \tag{2.9}$$

The results of this section resemble those of the stationary case, the difference being the existence of the term with $\partial/\partial T$ and $\partial/\partial T_1$, as we should expect from our time-dependent local form. By using

$$S(x, y; T, T_1)=S(x, y)\,\delta(T-T_1),$$

we see that Eqs. (2.8) and (2.9) are reduced to the stationary form, as in [1] [2].

It was noticed by Reid [5] that the stationary state form can be written as a single matrix Riccati equation. Our time-dependent state form also can be expressed as a single matrix equation, namely,

$$\frac{\partial}{\partial y}P+\frac{1}{\lambda}\frac{\partial}{\partial T}({}_0P)-\frac{1}{\lambda}\frac{\partial}{\partial T_1}(P_0)=A+B_1P+PB_2+PCP \tag{2.10}$$

for (2.8), or

$$-\frac{\partial}{\partial x}Q + \frac{1}{\lambda}\frac{\partial}{\partial T}({}_0Q) - \frac{1}{\lambda}\frac{\partial}{\partial T_1}(Q_0) = C + B_2Q + QB_1 + QAQ \tag{2.11}$$

for (2.9). Here, as in the work of Reid,

$$S = \begin{pmatrix} t & \rho \\ r & \tau \end{pmatrix}, \quad P = SJ, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} b_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}.$$

However, we must also introduce

$$P_0 = PJ_0, \quad {}_0P = J_0P, \quad Q_0 = QJ_0, \quad Q_0 = J_0Q$$

with

$$Q = JS, \quad J_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is noticed Eqs. (2.8) and (2.9) are derived from Eqs. (2.5). It is possible to proceed in the opposite direction and derive the local form from the state form. Hence two forms are equivalent.

3. Applications

3.1 *The Problem of Particles Moving in a Rod*

The physical model considered here is similar to the stationary case (p. 21, [2]), but the system now under consideration is time-dependent. The intensities as well as operators are functions of time. The collision phenomenon is dependent not only on location and on the original moving direction before collision, but also on the time at which the collision is taking place. We write $\sigma_\pm(z;\, T)\Delta, F_\pm(z;\, T)$, $G_\pm(z;\, T)$ and $I_\pm(z;\, T)$ to show they are time dependent, and correspond to the notation $\sigma_\pm(z)\Delta, F_\pm(z)$; $G_\pm(z)$, and $I_\pm(z)$ in the stationary case.

As in the stationary case, we compute the expected number of transmitted and reflected particles, by assuming there is always a time delay Δ/λ between input and output on a sufficiently thin layer, with thickness Δ. Then by our definition of time-dependent coefficients,

$$\begin{aligned} a(z;\, T, T_1) &= \{\sigma_-(z;\, T)G_-(z;\, T)\}\,\delta(T - T_1) \\ b_1(z;\, T, T_1) &= \{\sigma_+(z;\, T)[F_+(z;\, T) - 1]\}\,\delta(T - T_1) \\ b_2(z;\, T, T_1) &= \{\sigma_-(z;\, T)[F_-(z;\, T) - 1]\}\,\delta(T - T_1) \\ c(z;\, T, T_1) &= \{\sigma_+(z;\, T)G_+(z;\, T)\}\,\delta(T - T_1). \end{aligned} \tag{3.1.1}$$

It is readily apparent that the time-dependent local form is:

$$\begin{pmatrix} \left(\frac{\partial}{\partial z} + \frac{1}{\lambda}\right) I_+(z; T) \\ -\left(\frac{\partial}{\partial z} + \frac{1}{\lambda}\right) I_-(z; T) \end{pmatrix} = \begin{pmatrix} \sigma_+[F_+ - 1] & \sigma_- G_- \\ \sigma_+ G_+ & \sigma_-[F_- - 1] \end{pmatrix} \begin{pmatrix} I_+(z; T) \\ I_-(z; T) \end{pmatrix}. \tag{3.1.2}$$

By combining Eq. (2.8) and (2.9), the result is a set of equations for state forms, one of which is the right-hand reflection operator equation, namely,

$$\begin{aligned} \left(\frac{\partial}{\partial y} + \frac{1}{\lambda}\frac{\partial}{\partial T} - \frac{1}{\lambda}\frac{\partial}{\partial T_1}\right) \rho(x, y; T, T_1) &= \sigma_-(y; T) G_-(y; T)\,\delta(T - T_1) \\ &+ \sigma_+(y; T)[F_+(y; T) - 1]\,\delta(T - T_2)\rho(x, y; T_2 T_1) \\ &+ \rho(x, y; T, T_2)\sigma_-(y; T_2)[F_-(y; T_2) - 1]\,\delta(T_2 - T_1) \\ &+ \rho(x, y; T, T_3)\sigma_+(y; T_3) G_+(y; T_3)\,\delta(T_3 - T_2)\rho(x, y; T_2, T_1). \end{aligned} \tag{3.1.3}$$

We observe that this equation can be cast into the form which is identical to that of Wing's (p. 76 [6]), if we let the input of the rod (x, y), at y-end be $I_-(y; T_1) = \delta(T_1 - T_0)$, as $\sigma_+ = \sigma_- = \sigma$, $F_+ = F_- = F$, $G_+ = G_- = G$ and we define

$$R(x, y; T, T_0) = \lambda \int_{T_0}^{T} \rho(x, y; T, T_0)\, dT,$$

where λ is considered as a constant.

3.2 *Radiative Transfer in a Slab*

The problem of radiative transfer in a slab now is being extended to the time-dependent case, that is, the roles of $I(z; \pm\mu, \varphi_1)$, $\sigma(z)$ and $p(z; \mu; \mu_1, \varphi_1)$ (p. 37, [2]) are replaced by time-dependent functions $I(z; \pm\mu, \varphi_1; T)$, $\sigma_+(z; T)$ and $p(z; \mu, \varphi; \mu_1, \varphi_1; T)$, respectively.

To simplify our analysis, we assume for a slab $(z, z + \Delta)$, where Δ is sufficiently small, there exists a time delay of exactly Δ/λ between input and response. Our derivation for the coefficients is similar to the stationary case, attention, of course, being given to the time parameter. Details are omitted and we just present our results as follows.

Within $0(\Delta)$,

$$a(z; T, T_1) = \frac{1}{4\pi\mu} [\sigma(z; T)\bar{p}(z; \mu, \varphi; -\mu_1, \varphi_1; T)]\,\delta(T - T_1)$$

$$
\begin{aligned}
b_1(z; T, T_1) &= \left[\frac{1}{4\pi\mu}\,\sigma(z; T)\bar{p}(z; \mu, \varphi; \mu_1, \varphi_1; T)\right. \\
&\qquad \left. - \frac{1}{\mu_1}\,\sigma(z; T)\,\delta(\mu - \mu_1)\,\delta(\varphi - \varphi_1)\right]\delta(T - T_1) \\
b_2(z; T, T_1) &= \left[\frac{1}{4\pi\mu}\,\sigma(z; T)\bar{p}(z; -\mu, \varphi; -\mu_1, \varphi_1; T)\right. \\
&\qquad \left. - \frac{1}{\mu_1}\,\sigma(z; T)\,\delta(\mu - \mu_1)\,\delta(\varphi - \varphi_1)\right]\delta(T - T_1) \\
c(z; T, T_1) &= \frac{1}{4\pi\mu}\,\sigma(z; T)\bar{p}(z; -\mu, \varphi; \mu_1, \varphi_1; T)\,\delta(T - T_1),
\end{aligned}
\tag{3.2.1}
$$

where

$$
\bar{p}(z; \pm\mu, \varphi; \pm\mu_1, \varphi_1; T) = \frac{4\pi\mu}{\mu_1}\,p(z; \pm\mu, \varphi; \pm\mu_1, \varphi_1; T).
$$

Now, by substituting the above coefficients into our generalized time-dependent local form, as in the stationary case, we obtain a single equation,

$$
\begin{aligned}
\left(\frac{\partial}{\partial z} + \frac{1}{\lambda}\frac{\partial}{\partial T}\right) I(z; \mu, \varphi; T) &= -\frac{1}{\mu}\,\sigma(z; T) I(z; \mu, \varphi; T) \\
&+ \frac{1}{4\pi\mu}\int_{-1}^{1}\int_{0}^{2\pi} \sigma(z; T)\bar{p}(z; \mu, \varphi; \mu_1, \varphi_1; T) I(z; \mu_1, \varphi_1; T)\,d\mu_1\,d\varphi_1\,.
\end{aligned}
\tag{3.2.2}
$$

If we let $\lambda = c\mu$, where c may be called the propagation speed in the direction θ (recall $\mu = \cos\theta$), then (3.22) is identical to the results of Wing's (p. 72, [6]). However, Wing used an *ad hoc* approach based on particle counting, whereas our method obtains the result as part of a unified theory.

As for the time-dependent state form, we will consider the equations for right-hand reflection only. To make our result resemble the previously cited result of Chandrasekhar in the stationary case, we replace the right-hand reflection operator

$$
\rho(x, y; \mu, \varphi; -\mu_1, \varphi_1; T, T_1)
$$

by

$$
\frac{1}{4\pi\mu}\,S(x, y; \mu, \varphi; \mu_1, \varphi_1; T, T_1). \tag{3.2.3}
$$

The result is

$$\frac{1}{\sigma(y;T)}\left[\frac{\partial}{\partial y}+\frac{1}{\lambda}\left(\frac{\partial}{\partial T}-\frac{\partial}{\partial T_1}\right)+\left(\frac{1}{\mu}+\frac{1}{\mu_1}\right)\sigma(y;T)\right]$$
$$\times S(x,y;\mu,\varphi;\mu_1,\varphi_1;T,T_1)$$
$$=\bar{p}(y;\mu,\varphi;-\mu,\varphi;T)$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\bar{p}(y;\mu,\varphi;\mu_2,\varphi_2;T)\,S(x,y;\mu_2,\varphi_2;\mu_1,\varphi_1;T,T_1)\frac{d\mu_1}{\mu_1}\delta\varphi_1$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}S(x,y;\mu,\varphi;\mu_2,\varphi_2,T)\bar{p}(y;-\mu_2,\varphi_2;-\mu_1,\varphi_1,T,T_1)\frac{d\mu_2}{\mu_2}d\varphi_2$$
$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}S(x,y;\mu,\varphi;\mu,\varphi_3;T,T_2)\,\bar{p}(y;-\mu_3,\varphi_3;\mu_2,\varphi_2;T_2)$$
$$S(s,y;\mu_2,\varphi_2;\mu_1,\varphi_1;T_2,T_1)\frac{d\mu_1}{\mu_1}d\varphi_1\frac{d\mu_2}{\mu_2}d\varphi_2. \tag{3.2.4}$$

We note the function $\sigma(z;T)$ cannot be eliminated, as it is in the stationary case by introducing the optical depth, since σ is a function of time. However, if $\sigma=\sigma(z)$ is independent of time, then we are able to use the optical depth h, as defined by

$$\frac{dh}{dy}=\sigma(z).$$

In this case, we go through an analysis similar to that in the stationary case. The result is that the left side of (3 2.4) is replaced by

$$\left[\frac{\partial}{\partial h_1}+\frac{1}{\lambda}\left(\frac{\partial}{\partial T}-\frac{\partial}{\partial T_1}\right)-\left(\frac{1}{\mu}+\frac{1}{\mu_1}\right)\right]S(h_0,h_1;\mu,\varphi;\mu_1,\varphi_1;T,T_1)$$

and the right side of (3. 2.5) remains the same except that x is replaced by h_0 and y by h_1.

By using the optical depth and assuming that the scattering is isotropic (that is, $\bar{p}(h;\mu,\varphi;\mu_1,\varphi_1;T)=\gamma(h;T)$), we get the corresponding local form,

$$\mu\left(\frac{\partial}{\partial h}+\frac{1}{\lambda}\frac{\partial}{\partial T}\right)I(h;\mu,\varphi;T)=-I(z;\mu,\varphi;T)$$
$$+\frac{1}{4\pi}\gamma(h;T)\int_{-1}^1\int_0^2 I(z;\mu_1,\varphi_1;T)\,d\mu_1\,d\varphi_1. \tag{3.2.5}$$

In the case ρ and S are independent of time, equation (3. 2.3) is reduced to the result of Chandrasekhar (p. 169, [7]). By the corresponding local form, we obtain two equations, which are identical to each other, and they are usually called the *Boltzmann's transport equation* (for example, p. 4, [8]).

3.3 *Time-Dependent Photon Diffusion Problem*

The physical problem considered here is a time-dependent one-dimensional photon diffusion process. The specific intensity varies with position as well as time; for example, $I_+(z; T)$ denotes the right-moving intensity at position z and at time T, with $x \leqslant z \leqslant y$. The scattering of light in the section (x, y) is assumed to be isotropic.

We shall construct the time-dependent state form by using a technique similar to that used in [2]. That is, we shall obtain the time-dependent coefficients a, b_i, c by comparison of our generalized local form with the known differential equation for photon diffusion.

The equation of transfer corresponding to the present case is given by [9],

$$\frac{\partial I_+(z; T)}{\partial z} + \frac{1}{\lambda}\frac{\partial I_+(z; T)}{\partial T} = -l(z; T)I_+(z; T) + B(z; T)$$

and

$$\frac{\partial I_-(z; T)}{\partial z} - \frac{1}{\lambda}\frac{\partial I_-(z; T)}{\partial T} = l(z; T)I_-(z; T) - B(z; T), \tag{3.3.1}$$

where λ is the speed of light, $l(z; T)$ is the extinction coefficient and $B(z; T)$ is the source function; that is,

$$B(z; T) = \frac{1}{2}\,\sigma(z; T)\int_{T_0}^{T} [I_+(z; T_1) + I_-(z; T_1)]\,\frac{1}{d}\exp\left(-\frac{T - T_1}{d}\right) dT_1,$$

where $\sigma(z; T)$ is the scattering coefficient and d is the duration of temporal capture. By comparing (3. 3.1) with our generalized local form, we get

$$\begin{aligned} b_1(z; T, T_1) = b_2(a; T, T_1) &= -l(z; T)\,\delta(T - T_1) \\ &\quad + \frac{\sigma(z; T)}{2d}\exp\left(-\frac{T - T_1}{d}\right) \\ a(z; T, T_1) = c(z; T, T_1) &= \frac{\sigma(z; T)}{2d}\exp\left(-\frac{T - T_1}{d}\right) \end{aligned} \tag{3.3.2}$$

and $a = b_1 = b_2 = c = 0$ when $T < T_1$.

Now by our generalized time-dependent state form for right-hand reflection, we get the results of Bellman *et al.* [10].

3.4 *Time-Dependent Homogeneous Diffusion Process*

Now, we consider two-dimensional time-dependent diffusion processes. The transmissions and reflections take place in an obstacle extended from $z = x$ to $z = y$. Besides z, there is a second space variable h, $-\infty \leqslant h \leqslant \infty$, the h axis being perpendicular to the z axis.

Let $I_+(x, h_1; T_1)$ be the right-moving incident intensity impinging on the obstacle at $h = h_1$ and at time T_1. This incident intensity will produce a transmitted right-moving intensity at position h and time T, and is denoted by

$$\begin{aligned} I_+(x, h; T) &= [t(x, y) I_+(x)](h; T) \\ &= \int_{-\infty}^{\infty} t(x, y; h, h_1; T, T_1)\, I_+(x, h_1; T_1)\, dh_1, \end{aligned} \qquad (3.4.1)$$

where the operator $t(x, y)$ is represented by a continuous function $t(x, y; h_1; T, T_1)$ and the summation convention is used. For the homogeneous case,

$$t(x, y; h, h; T, T_1) = t(x, y; h - h_1, T - T_1). \qquad (3.4.2)$$

This particular form indicates that there is spatial and temporal homogeneity, in that t is not dependent upon the absolute time T and absolute location h. However, homogeneity in the z direction is not assumed; t depends on (x, y) rather than just on y-x.

We use corresponding notations and assume similar homogeneity properties for the operators, τ, ρ, and r.

We introduce the Laplace transform L_p and L_q defined by

$$L_p t(x, y; h; T) = \int_{-\infty}^{\infty} t(x, y; h; T)\, e^{-ph}\, dh$$

$$L_q t(x, y; h; T) = \int_{-\infty}^{\infty} t(x, y; h; T)\, e^{-qT}\, dT.$$

For brevity, we write $L_p L_q = L, Lt(x, y; h; T) = \Lambda(p, q; t)$. Then by repeated application of the convolution theorem, we have

$$\Lambda(p, q; t) L I_+(x) = Lt(x, y)\, I_+(x). \qquad (3.4.3)$$

That is, the operator $t(x, y)$ admits the representation

$$t(x, y) = L^{-1}\, \Lambda(p, q; t) L. \qquad (3.4.4)$$

In a similar manner, we do the same for the other operators. Now we take double Laplace transform L on both sides of our generalized time-dependent state form (2.8) and use properties of our homogeneous operators (3. 4.4). Details of computation are not presented here, but results are stated as

$$\frac{\partial}{\partial y} \begin{pmatrix} \underline{t} & \underline{\rho} \\ \underline{r} & \underline{\tau} \end{pmatrix} - \frac{q}{\lambda} \begin{pmatrix} 0 & \underline{\rho} \\ 0 & \underline{\tau} \end{pmatrix} = \begin{pmatrix} \underline{b_1 t} + \underline{\rho c t} & \underline{a} + \underline{b_1 \rho} + \underline{\rho b_2} + \underline{\rho c \rho} \\ \underline{\tau c t} & \underline{\tau b_2} + \underline{\tau c \rho} \end{pmatrix}, \qquad (3.4.5)$$

where $\underline{f} = \Lambda(p, q, f)$.

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